$$\alpha\Phi^* (T_0)\, \varphi_0 = \beta\Phi^* (T_1)\, \varphi_1 \tag{7.4}$$

Here $\alpha = | \Phi^* (T_1)\, \varphi_1 |$, $\beta = | \Phi^* (T_0)\, \varphi_0 |$. Then multiplying (7.4) scalarly by $u\,(\varphi_0)$ and $u\,(\varphi_1)$, we obtain

$$\alpha f\,(T_0) = \beta f\,(T_1)\, (\varphi_0 \cdot \varphi_1), \qquad \alpha f\,(T_0)\, (\varphi_0 \cdot \varphi_1) = \beta f\,(T_1)$$

Hence $(\varphi_0 \cdot \varphi_1) = 1$ and so $\varphi_1 = \varphi_0$. A contradiction.

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IMPULSE TRACKING OF A POINT WITH BOUNDED THRUST

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We examine the game problem [1 – 3] of the contact of two material points with unit masses moving in a three-dimensional space under the action of only the controlling forces F_1 and F_2 arbitrary in direction. It is assumed that force F_1 is bounded in momentum, while F_2, in absolute value. In parallel we consider two problems on the minimax time up to "hard" (with respect to the coordinates) and up to "soft" (with respect to the coordinates and velocities) contact. In both problems the whole space of positions is divided into two regions. The optimal controls of the first (the minimizing) player (point) and of the second (the maximizing) player (point) are formed in the first region and the minimax time computed as well. The second player's control permitting him to evade contact under any action of the first player is formed in the second region. A comparison is made with a previously-considered case [4] in which both points can move along certain fixed straight lines.

1. Let the equations of motion have the form

$$x^{\cdot} = y, \qquad y^{\cdot} = u + v \tag{1.1}$$

$$\mu - \int_0^{\tau} |u|\, dt = \mu^{(1)}(\tau) \geqslant 0 \tag{1.2}$$

$$|v| \leqslant 1 \tag{1.3}$$

$$y^{(1)} = y + \mu_1, \qquad \mu^{(1)} = \mu - |\mu_1| \geqslant 0 \tag{1.4}$$

Relations (1.1) – (1.4), in which x, y, u, v, μ_1 are three-dimensional vectors and μ, $\mu^{(1)}$ are nonnegative numbers, can be interpreted as equations of the relative $(r_1 - r_2 = x,\ r_1^{\cdot} - r_2^{\cdot} = y)$ motion of two points with masses m_1 and m_2 whose positions and velocities relative to some fixed center are given by the vectors (r_1, r_2) and $(r_1^{\cdot}, r_2^{\cdot})$, respectively. It is assumed that the points are subject to the action of only the controlling forces F_1 and F_2, arbitrary in direction and restricted by constraints (1.2), (1.3) with $u = F_1 / m_1$ and $v = -F_2 / m_2$. Constraint (1.2) allows the instantaneous variation of vector y and of number μ by formulas (1.4), where μ_1 is a finite vector with Euclidean modulus $|\mu_1|$. In this case the control $u = \mu_1 \delta$ is called an impulse control.

The vector $w = [x, y, \mu]$ is called the position, while the result of the impulse actions of the first point (player) is denoted by

$$w^{(1)} = [x,\ y^{(1)} = y + \mu_1(w),\ \mu^{(1)} = \mu - |\mu_1(w)|]$$

If vector $w^{(1)}(t \geqslant 0)$ is specified as a function of time, then the vector $w^{(1)}(t - 0)$ is named the position $w(t > 0)$, while for $t = 0$ we take $w(0) = [x(0),\ y(0),\ \mu(0)]$. The pair of controls $u(w, v)$, $v(w)$ and the unique trajectory $w^{(1)}(t \geqslant 0,\ \{u(w, v), v(w)\},\ w(0))$ corresponding to them are said to be admissible if constraints (1.2), (1.3) are fulfilled for all t and Eqs. (1.1) are satisfied for almost all t. Furthermore, the trajectory is right-continuous for all t admits of a finite number of jumps on every finite interval $0 \leqslant t \leqslant t_1$, consistent with (1.4), and is absolutely continuous on the intervals of continuity.

On the admissible trajectories we consider a game with the termination set (the contact set)

$$M\ [|x| = 0,\ k(\mu - |y|) \geqslant 0,\ k = 0,1]$$

For $k = 0$ the velocity y is arbitrary and the contact is termed hard. For $k = 1$ the first player can instantaneously null his velocity and the contact is soft. At every possible position the players solve one of two problems.

Problem 1. Find a pair $u^{\circ}(w, v), v^{\circ}(w)$ such that the time $T[u(w, v), v(w)]$ of first hitting of a position onto set M satisfies the bounds

$$T[u^{\circ}, v] \leqslant T[u^{\circ}, v^{\circ}] \leqslant T[u, v^{\circ}]$$

Problem 2. Find a control $v_0(w)$ such that under any control $u(w, v)$ the position would not hit upon M in finite time.

The collections of positions admitting of a solution of Problem 1 and Problem 2 are denoted $W^0(M)$ and $W_0(M)$. By y_α, y_β we denote the projections of vector y onto vector x and onto a plane perpendicular to vector x, and we choose a right-hand triple of unit vectors j_α, j_β, j_γ according to the formulas

$$j_\alpha = x / |x|, \quad j_\beta = y_\beta / |y_\beta| \quad \text{for } w \in D_1[|x| > 0, |y_\beta| > 0]$$
$$j_\alpha = x / |x|; \; j_\beta, j_\gamma \quad \text{are arbitrary for} \quad w \in D_2[|x| > 0, |y_\beta| = 0]$$
$$j_\alpha, j_\beta, j_\beta \text{ are arbitrary for } w \in D_3 [\, |x| = 0]$$

Denoting the components of a three-dimensional vector along the indicated unit vectors by indices α, β, γ, as a consequence of Eqs. (1.1), (1.2), (1.4) we obtain

$$|x|^\cdot = y_\alpha, \qquad y_\alpha^\cdot = u_\alpha + v_\alpha + y_\beta^2 / |x| \qquad (1.5)$$
$$|y_\beta|^\cdot = u_\beta + v_\beta - y_\alpha |y_\beta| / |x|$$

$$|x|^\cdot = y_\alpha, \quad y_\alpha^\cdot = u_\alpha + v_\alpha, \quad |y_\beta|^\cdot = \sqrt{(u_\beta + v_\beta)^2 + (u_\gamma + v_\gamma)^2} \qquad (1.6)$$

$$|x|^\cdot = |y| \qquad (1.7)$$

$$y_\alpha^{(1)} = y_\alpha + \mu_{1,\alpha}, \qquad |y_\beta^{(1)}| = \sqrt{(|y_\beta| + \mu_{1,\beta})^2 + \mu_{1,\gamma}^2} \qquad (1.8)$$

Equations (1.5) – (1.7) follow from Eqs. (1.1) for $w \in D_1$, D_2, D_3, respectively, while Eq. (1.8) follows from (1.4) for $w \in D_1 \cup D_2$. It is clear that the solution depends solely on the quantities $|x|$, y_α, $|y_\beta|$, μ, k; we retain the notation w for this collection.

2. We note one simple property of the problem. Let there exist a function $\nu(w)$ satisfying the bounds

$$y_\alpha + \nu < 0 \qquad (2.1)$$

$$q_1(w, \nu) = \mu + k(y_\alpha + \nu) - \sqrt{y_\beta^2 + \nu^2} + |x| / (y_\alpha + \nu) \geqslant 0$$

and such that from the equality

$$\nu(|x|, y < 0, \; |y_\beta| = 0, \mu, k) = 0 \qquad (2.2)$$

there follows, for any $|x_2| \leqslant |x|$, $\mu_2 \leqslant \mu$, the equality

$$\nu(|x_2|, y_\alpha < 0, |y_\beta| = 0, \mu_2, k) = 0 \qquad (2.3)$$

then the control

$$u_\nu = \nu(w)\,\delta j_\alpha - |y_\beta|\,\delta j_\beta, \quad w \in [y_\beta^2 + \nu^2 > 0]$$
$$u_\nu = -v, \qquad w \in [y_\beta^2 + \nu^2 = 0]$$

leads the position onto set M in the time

$$T[u_\nu, v] = -|x| / (y_\alpha + \nu(w))$$

Indeed, the control $u_\nu = -v$ leads the trajectory away from the vector

$$w^{(1)}(0) = [|x(0)|, y_\alpha^{(1)}(0) < 0, |y_\beta^{(1)}(0)| = 0, \mu^{(1)}(0) > 0, k]$$

in accordance with the equations

$$|x|^\cdot = y_\alpha^{(1)}(0), \; y_\alpha^\cdot = |y_\beta|^\cdot = 0, \quad \mu^\cdot = -|v|, \quad \nu^\cdot = 0 \qquad (2.4)$$

The bounds $y_\alpha^{(1)}(0) < 0$, $\mu^{(1)}(0) \geqslant 0$ follow from bounds (2.1). The derivative $q_1^\cdot(w, \nu) = 1 - |v| \geqslant 0$ is nonnegative and, consequently, the bound $q_1(w, \nu) \geqslant 0$ is not violated for a motion by virtue of Eqs. (2.4), while the validity of the equation $\nu^{\cdot} = 0$ follows from (2.2), (2.3). Equations (2.4) lead the position onto the set $|x| = 0$ with a preservation of the bound $\mu + ky_\alpha = \mu - k|y| \geqslant 0$. This signifies that the inclusion $w(t_1) \in M$ is effected at the instant $t_1 = -x(0) / (y_\alpha^{(1)}(0)) = -|x| / (y_\alpha + \nu)$.

The reasonings presented bring us naturally to the investigation of the maximum of the function

$$q(w, p) = \mu + kp - \sqrt{y_\beta^2 + (p - y_\alpha)^2} + |x| / p$$

with respect to the variable p in the region $p < 0$. Computing the first $q^{(1)}$ and the second $q^{(2)}$ derivatives with respect to p for fixed w, for $w \in D_1$ we have

$$\begin{gathered} q^{(1)} = -(p - y_\alpha) / l(w, p) + k - |x| / p^2 \\ \lim q^{(1)} = -\infty, \qquad p \to -0 \\ \lim q^{(1)} = 1 + k > 0, \qquad p \to -\infty \\ q^{(2)} = -y_\beta^2 / l^3(w, p) + 2|x| / p^3 < 0 \\ l(w, p) = \sqrt{y_\beta^2 + (p - y_\alpha)^2} \end{gathered} \tag{2.5}$$

In region D_2 we have

$$q^{(1)} = \begin{cases} -1 + k - |x| / p^2, & p - y_\alpha > 0 \\ 1 + k - |x| / p^2, & p - y_\alpha < 0 \end{cases} \tag{2.6}$$

Formulas (2.5), (2.6) show that for $w \in D_1 \cup D_2$ there exists a unique function $p_1(w)$ which for $w \in D_1$ is a solution of the equation $q^{(1)}(w, p) = 0$, while for $w \in D_2$ is given by the formulas

$$\begin{gathered} p_1(w) = -\sqrt{|x| / (1 + k)}, \qquad w \in F = D_2 \cap [a(w) = \\ -\sqrt{|x| / (1 + k)} - y_\alpha < 0] \\ p_1(w) = y_\alpha, \qquad w \in E = D_2 \cap [a(w) \geqslant 0] \end{gathered} \tag{2.7}$$

This function realizes the maximum of the function $q(w, p)$ on the variable p in the region $p < 0$. Denoting this maximum by $r(w)$, we have

$$r(w) = q(w, p_1(w)) = \max_{p<0} q_1(w, p), \qquad w \in D_1 \cup D_2$$

We prolong the function $r(w)$ onto the set D_0 [$|x| = 0$, $k(\mu - |y|) < 0$, $k = 1$] by the formula

$$r(w) = \mu - |y|, \qquad w \in D_0$$

Leaving aside for the present a consideration of the sets D_0, M, we prove a lemma in the region $D_1 \cup D_2$.

Lemma 2.1. Any impulse control $u = \mu_1 \delta$ realizes a nonpositive increment

$$\Delta q = q(w^{(1)}, p) - q(w, p) \leqslant 0 \tag{2.8}$$

and bound (2.8) becomes an equality only on the family of controls

$$\begin{gathered} u(w, p, m) = m(p - y_\alpha)\delta j_\alpha - m |y_\beta| \delta j_\beta \\ [0 \leqslant m \leqslant \lambda_1(w, p) = \min [1, \mu / l(w, p)] \end{gathered}$$

Proof. Let us assume to the contrary that

$$\Delta q = l(w, p) - |\mu_1| - l(w^{(1)}, p) > 0 \tag{2.9}$$

From bound (2.9) follow the bounds

$$l(w, p) - |\mu_1| \geqslant 0 \tag{2.10}$$

$$(l(w, p) - |\mu_1|)^2 - l^2(w^{(1)}, p) = \tag{2.11}$$
$$-2(|y_\beta|\mu_{1,\beta} + (p - y_\alpha)\mu_{1,\alpha} + |\mu_1| l(w, p)) > 0$$

Bound (2.11) is violated for any vector μ_1. The contradiction proves bound (2.8). The equality $\Delta q = 0$ implies bound (2.10) and the equality to zero of the left-hand side of bound (2.11). From this equality it follows that vectors μ_1 and $u(w, p, 1)$ should be proportional $\mu_1 = nu(w, p, 1)$; $n \geqslant 0$. If the number $\lambda_1(w, p) < 1$, then the maximal admissible value of $|\mu_1|$ is the number μ. If, however, $\mu / l(w, p) > 1$, then bound (2.10) is violated when $\mu_1 > \dot{l}$. This proves that bound (2.8) becomes an equality on the controls of the family $u(w, p, m)$ and only on them.

Lemma 2.1 permits us to obtain a corollary.

Corollary 2.1. Any impulse control $u = \mu_1\delta$ realizes a nonpositive increment

$$\Delta r = r(w^{(1)}) - r(w) \leqslant 0 \tag{2.12}$$

and bound (2.12) becomes an equality only on the family of controls

$$u_1(w, p_1(w), m) = m(p_1(w) - y_\alpha)\delta j_\alpha - m|y_\beta|\delta j_\beta$$
$$0 \leqslant m \leqslant \lambda_1(w, p_1(w)) \tag{2.13}$$

Bound (2.12) is a simple consequence of Lemma 2.1. Let us prove that bound (2.12) becomes an equality on family (2.13). By Lemma 2.1 we have the bound $q(w^{(1)}(w, u_1), p) \leqslant q(w^{(1)}(w, u_1), p_1(w))$, it turns into an equality for $p = p_1(w)$. This signifies that $p_1(w^{(1)}(w, u_1)) = p_1(w)$. The last equality proves that (2.12) becomes an equality on family (2.13) and only on it.

3. Let us compute the right time derivative of the function $r(w)$ relative to Eqs. (1.5) in collection with the equation $\mu^{\cdot} = -|u|$, by separating the derivative into two terms

$$r^{\cdot}(w) = R_1(w, p_1) + R_2(w, p_1, u, v)$$

where $p_1 = p_1(w)$ and the term $R_2(w, p_1, u, v)$ is annihilated when $u = v = 0$. We carry out the computations at first for $w \in D_1 \cup F$ because the function $l(w, p_1)$ does not vanish in this region and the computations can be carried out by the rule of differentiating the implicit function $p_1(w)$ defined by the equation $q^{(1)}(w, p) = 0$.

According to the indicated arguments we have

$$R_1(w, p_1) = p_1 y_\beta^2 / |x| l(w, p_1) + y_\alpha / p_1 - 1 + 1 =$$
$$p_1 / |x[\,{}'y_\beta^2 / l(w, p_1) - (p_1 - y\alpha)|x| / p_1^2] + 1$$

If in the second term with the brackets we replace the factor $|x| / p_1^2$ by the expression $(p_1 - y_\alpha) / l(w, p_1) + k$, from the equation $q^{(1)}(w, p) = 0$ we obtain

$$R_1(w, p_1(w)) = (p_1 / |x|)(l(w, p_1) - k(p_1 - y_\alpha)) + 1 < 1 \tag{3.1}$$

Bound (3.1) is a corollary of bound $p_1(w) < 0$. The term R_2 has the form

$$R_2(w, p_1, u, v) = -|u| + ((p_1 - y_\alpha)(u_\alpha + v_\alpha) - |y_\beta|(u_\beta + v_\beta)) / l(w, p_1) \qquad (3.2)$$

The computation of the derivative $r^{\cdot}(w)$ for $w \in E$ requires details which at the same time prove the continuity of the function $p_1(w)$ for $w \in E$. The substitution $z = p - y_\alpha$ reduces the equation $q^{(1)} = 0$ to the form

$$z(y_\alpha + z) + (|x| - k(y_\alpha + z)^2)\sqrt{y_\beta^2 + z^2} = 0 \qquad (3.3)$$

while the substitution $|x| = (k - s)y_\alpha^2$ reduces (3.3) to the form

$$z(y_\alpha + z)^2 + (-sy_\alpha^2 - 2ky_\alpha z - kz^2)\sqrt{y_\beta^2 + z^2} = 0 \qquad (3.4)$$

We carry out the examination for small $|y_\beta| > 0$ in neighborhoods of regions E_1, E_2, E_3 separated from region E by the relations $[s = 0]$, $[-1 < s < 0;\ 0 < s < 1]$, $[s = -1]$, where $s(w) = k - |x| / y_\alpha^2$.
In the neighborhood of region E_1 we employ the substitution $z = Zs|y_\beta|$ which gives Eq. (3.4) the form

$$s|y_\beta|((Z - 1)y_\alpha^2 + P(w, Z)) = 0 \qquad (3.5)$$

We denote the collection of s, y_α, $|y_\beta|$, k by w. The function $P(w, z)$ is continuous and vanishes for $s|y_\beta| = 0$. The form of Eq. (3.5) shows that in the neighborhood of region E_1 the solution $z(w)$ of Eq. (3.3) can be represented in the form

$$z_1(w) = Z_1(w)s|y_\beta|, \quad s = k - |x| / y_\alpha^2 \qquad (3.6)$$

Here $Z_1(w)$ is a positive bounded function which nulls the third factor in the left-hand side of Eq. (3.5).

In a neighborhood of regions E_2, E_3 we study a corollary of Eq. (3.3) taken in the form

$$z^2(y_\alpha + z)^4 - (|x| - k(y_\alpha + z)^2)^2(y_\beta^2 + z^2) = 0 \qquad (3.7)$$

In a neighborhood of region E_2 the substitution

$$|x| = y_\alpha^2(k - s), \qquad z = (s/\sqrt{1 - s^2} + Z)|y_\beta|$$

takes (3.7) to the form

$$P_2 + P_1 Z + P_0 = 0 \qquad (3.8)$$

Here P_2 is a polynomial in Z of degree not less than second, with continuous coefficients; P_1, P_0 are continuous and do not depend on Z; P_1 does not vanish for $|y_\beta| = 0$, $s \neq 0$, while P_0 does vanish for $|y_\beta| = 0$. Denoting by $Z_2(w)$ the continuous solution of Eq. (3.8), vanishing for $|y_\beta| = 0$, we obtain the solution of Eq. (3.7) in the form

$$z_1(w) = (s/\sqrt{1 - s^2} + Z_2(w))|y_\beta| \qquad (3.9)$$

In the neighborhood of region E_3 it is convenient to present Eq. (3.7) in the form

$$\psi(w, z) = \psi_1(w, z) + \psi_2(w, z) = 0$$

$$\psi_1(w, z) = z^2(z - \lambda)(2y_\alpha + \lambda + z)[(y_\alpha + z)^2(1 - k) + (\lambda + y_\alpha)^2(1 + k)]$$

$$\psi_2(w, z) = -[(\lambda + y_\alpha)^2(1 + k) - k(y_\alpha + z)^2]^2 |y_\beta|^2$$

Here λ denotes the function $-\sqrt{|x|/(1+k)} - y_\alpha$, equal to zero when $s = k - |x|/y_\alpha^2 = -1$.

The substitution

$$z = Z y_\beta^{2/3} + {}^1/_2 (\lambda - |\lambda|)$$

permits us to obtain, for $Z \leqslant 0$, the bound

$$\psi(w, z) = \psi_3(w, Z) \geqslant y_\beta^2 [(1 + k) 4y_\alpha^3 Z^3 - 4y_\alpha^4 + P_3(w, Z)] \quad (3.10)$$

turning into an exact equality when $|y_\beta| Z = 0$. The function $P_3(w, Z)$ depends continuously on w, Z and vanishes when $|y_\beta| = \lambda = 0$. Bound (3.10) shows that the equation $\psi_3(w, Z) = 0$ admits of a strictly negative bounded solution $Z_3(w) < 0$ for $|y_\beta| > 0$. This means that the solution of Eq. (3.7) can be represented in the form

$$z_1(w) = Z_3(w) |y_\beta|^{2/3} + {}^1/_2 (\lambda(w) - |\lambda(w)|) < 0 \quad (3.11)$$

in the neighborhood of region E_3.

It is not difficult to verify that the function $z_1(w)$ given by formulas (3.9), (3.11) is a solution of Eq. (3.3). Indeed, from Eq. (3.7) follows either Eq. (3.3) or the equation

$$z(y_\alpha + z)^2 + (sy_\alpha^2 + 2ky_\alpha^2 + kz^2)\sqrt{y_\beta^2 + z^2} = 0 \quad (3.12)$$

However, the substitution of expressions (3.9), (3.11) into the left-hand side of (3.12) shows that Eq. (3.12) is automatically violated for sufficiently small $|y_\beta|$. The application of equality (3.9) permits us to express the derivative $r^\bullet(w)$ in the form

$$r^\bullet(w) = 1 - |u| + s(u_\alpha + v_\alpha) - \sqrt{1 - s^2}\sqrt{(u_\beta + v_\beta)^2 + (u_\gamma + v_\gamma)^2} \quad (3.13)$$

in region E_2. For $w \in E_3$ the increment Δr of function $r(w)$ can be represented as

$$\Delta r = \Delta |x| / y_\alpha + \Delta\mu + s\Delta y_\alpha + P(w, \Delta w, z(w + \Delta w))$$

where the function P does not contain terms of dimensions less than two in the increments Δx, Δy_α, $\Delta |y_\beta|$, $\Delta z(w) = z(w + \Delta w)$. A division by Δt and a passing to the limit as $\Delta t \to 0$ yield, according to (3.11), the equation

$$z^\bullet(w) = 1 - |u| + s(u_\alpha + v_\alpha)$$

It shows that equality (3.13) is valid also for $s = -1$. Analogous calculations show the validity of equality (3.13) also for $s = 0$.

4. The relations obtained above allow us to construct the control

$$v_1(w) = -[(p_1 - y_\alpha) j_\alpha - |y_\beta| j_\beta] / l(w, p_1), \quad w \in D_1 \cup F \quad (4.1)$$

$$v_1(w) = -s(w) j_\alpha + \sqrt{1 - s^2(w)}(\cos\varphi j_\beta + \sin\varphi j_\gamma), \quad w \in E \quad (4.2)$$

$$v_1(w) = y / |y|, \quad w \in D_0, \quad [|x| = 0, \ \mu - |y| < 0, \ k = 1] \quad (4.3)$$

where the angle φ in formula (4.2) is arbitrary, and to prove a theorem.

Theorem 4.1. Any admissible pair $u(w, v)$, $v_1(w)$ cannot lead the trajectory from the region $M_0 = [|x| > 0; \ r(w) < 0] \cup D$ onto set M in finite time.

Proof. We assume to the contrary that $w(0) \in M_0$ and $x(0) > 0$, but that the equality $x(\tau) = 0$ is fulfilled for some finite $\tau > 0$. Since the trajectory has only a finite number of jumps, we have

$$|x(\tau)|^{\cdot} = -|y(\tau)|, \quad |y_\beta((\tau)| = 0 \tag{4.4}$$

The equation $q^{(1)}(w, p) = 0$ and formula (2.8) permit us to obtain the bound $p_1(w) < -\sqrt{|x| / (1+k)}$ from which follows the equality

$$\lim (|x| / p_1(w)) = 0, \quad t \to \tau \tag{4.5}$$

Moreover, the relations

$$\lim (p_1(w(t)) - y_\alpha(t)) = 0, \quad t \to \tau, \quad k = 0$$
$$p_1(w(t)) - y_\alpha(t) \geqslant 0, \quad t \to \tau, \quad k = 1 \tag{4.6}$$

are valid. Indeed, when $y_\alpha(\tau) < 0$ and $k = 0$, $\lim s(w) = \lim -|x| / y_\alpha^2 = 0$ as $t \to \tau$, and formula (3.6) establishes the first equality. The equality $\lim s(w) = 1$ as $t \to \tau$ is valid when $k = 1$, $y_\alpha(\tau) < 0$, and formula (3.9) establishes the second relation in system (4.6). Furthermore, Corollary 2.2 and formulas (3.1), (3.2), (3.13) permit us to establish that the function $r(w)$ does not grow along any admissible trajectory corresponding to the pair $u(w, v)$, $v_1(w)$. Equalities (4.4), (4.5), and relations (4.6) permit us to establish the bound

$$\lim r(w(\tau)) = \mu(\tau) - k|y(\tau)| < 0, \quad t \to \tau$$

(it can be established for $y_\alpha(\tau) = 0$ by contradiction). This bound establishes the continuity of $r(w)$ for $w \in D_0$ and shows that the phase constraint $\mu \geqslant 0$ is violated when $k = 0$, while the inclusion $w(\tau) \in D_0$ is valid when $k = 1$. The passage from region D_0 with $v = v_1(w)$ is possible only in the region $[|x| > 0, r(w) < 0]$. The proof is completed.

The preceding results naturally lead to a consideration of the function $p_2(w)$, namely, the smallest root of the equation $q(w, p) = 0$ in the region M^0 $[r(w) \geqslant 0; |x| > 0]$, of the controls

$$\left.\begin{aligned} u_2(w) &= (p_2 - y_\alpha)\delta j_\alpha - |y_\beta|\delta j_\beta \\ v_2(w) &= -u_2(w) / |u_2(w)| \end{aligned}\right\} w \in (D_1 \cup F) \cap [r(w) \geqslant 0] \tag{4.7}$$

$$\left.\begin{aligned} u_2(w, v) &= -v \\ v_2(w) &= v_1(w) \end{aligned}\right\} \quad w \in E \cap [r(w) = 0] \tag{4.8}$$

(the unit vector $v_2(w)$ is antiparallel to the vector $u_2(w)$), as well as of the function

$$T(w) = -|x| / p_2(w); \quad w \in (D_1 \cup F \cup E) \cap [r(w) \geqslant 0]$$

$$T(w) = 0, \; w \in M$$

The derivative $(T(w))^{\cdot}$ has the form

$$(T(w))^{\cdot} = -1 - (l - k(p_2 - y_\alpha)) / (p_2 q^{(1)}) -$$
$$|x|(R_2(w, u, v) + 1) / (p_2^2 q^{(1)}) \tag{4.9}$$

for $w \in (D_1 \cup F) \cap [r(w) > 0]$. Here

$$q^{(1)} = q^{(1)}(w, p_2) = -(p_2 - y_\alpha) / l + k - |x| \; p_2^2 > 0$$

$$p_2 = p_2(w) < 0$$

$$l = l(w, p_2) = \sqrt{y_\beta^2 + (p_2 - y_\alpha)^2} \tag{4.10}$$

$$R_2(w, u, v) = -|u| + [(p_2 - y_\alpha)(u_\alpha + v_\alpha) - |y_\beta|(u_\beta + v_\beta)]\, l^{-1}$$

The bound $q^{(1)} > 0$ follows from the definition of function $p_2(w)$ as the smallest root of the equation $q(w, p) = 0$, while the bound $p_2(w) < 0$ follows from the obvious inequality $p_2(w) \leqslant p_1(w) < 0$. The bound

$$T^{\cdot}(w, u(w, v), v_2(w)) > -1 \quad \text{for} \quad w \in (D_1 \cup F) \cap [r(w) > 0] \qquad (4.11)$$

follows from formulas (4.9), (4.10). Lemma 2.1 allows us to deduce –

Corollary 4.1. Any impulse control $u = \mu_1 \delta$ preserving the inclusion $w^{(1)}(w, u) \in M^0$ realizes the nonnegative increment

$$\Delta T = T(w^{(1)}) - T(w) \geqslant 0 \qquad (4.12)$$

and bound (4.12) becomes a strict equality only on the family of controls $mw_2(w)$, $0 \leqslant m \leqslant 1$.

Proof. By Lemma 2.1 we have $q(w^{(1)}, p) \leqslant q(w, p)$ and the bound

$$q(w^{(1)}, p_2(w)) < q(w, p_2(w)) = 0$$

is valid when $u \neq mu_2(w)$. From this bound follow the bounds

$$p_2(w^{(1)}) > p_2(w), \qquad \Delta T > 0$$

If, however, the control is taken from the family $mu_2(w)$, then by the lemma, $q(w^{(1)}, p) < q(w, p < p_2(w))^2$ and $q(w^{(1)}, p_2(w)) = q(w, p_2(w)) = 0$. This means that $p_2(w^{(1)}) = p_2(w)$ and $\Delta T = 0$.

In the region $[\,|x| > 0,\ r(w) = 0]$, Lemma 2.1 and formulas (3.1), (3.2), (3.13) show that any control $u \neq u_2(w, v)$ in pair with $v_2(w)$ leads the position into the region $[\,|x| > 0,\ r(w) < 0]$. The only exception is the position $w \in E_3 \cap [r(w) = 0]$ at which any control

$$u^* = u_\alpha j_\alpha, \quad u_\alpha \leqslant 0$$

preserves the equality $r^{\cdot}(w) = 0$, while the derivative $(T(\omega))^{\cdot}$ has the form

$$(T(w))^{\cdot} = \begin{cases} -\,^1/_2, & -1 + \,^1/_2(k+1) < u_\alpha \leqslant 0 \\ -1 + (u_\alpha + 1)\,|x| / y_\alpha^2, & u_\alpha \leqslant -1 + \,^1/_2(k+1) \end{cases}$$

The bound $u_\alpha \leqslant -1 + \,^1/_2(k+1)$ is equivalent to the bound $s^{\cdot}(w, u^*, v_2(w)) \leqslant 0$. This signifies that from the bound $(T(w))^{\cdot} < -1$ there follow the bounds $u_\alpha + 1 < 0$, $s^{\cdot}(w, u^*, v_2(w)) < 0$ which show that any finite control u^* realizing the inequality $(T(w))^{\cdot} < -1$, instantaneously leads the position into region E_2. However, at this position impulse controls $u = \mu_1\delta$ are inadmissible since they realize the inclusion $w^{(1)} \in [r(w) < 0]$. The continuity of the functions $p_2(w)$, $T(w)$ and $p_1(w)$ in the region $[\,|x| > 0, r(w) > 0]$ follows from the implicit function theorem.

In the region $[\,|x| > 0, r(w) = 0]$ we assume to the contrary that there exists a sequence $w_i \to w$ as $t \to \infty$ such that the sequence $p_2(w_i)$ does not converge to $p_2(w)$. Obviously, it cannot be unbounded and cannot have $p = 0$ as its limit point, since $q(w, p) \to -\infty$ as $p \to -0, -\infty$. Consequently, p_3 exists; $p_2(w) \neq p_3 < 0$ is the limit point of the sequence $p_2(w_i)$, and $q(w, p_3) = 0$ as a consequence of the continuity of the function $q(w, p)$. The equalities $r(w) = q(w, p_3) = 0$, $p_1(w) = p_2(w)$ contradict the bound $r(w) > q(w, p)$ for $0 > p \neq p_1(w)$.

We note that by an analogous reasoning we can independently prove the continuity of the functions $p_1(w)$, $r(w)$ for $|x| > 0$. Let $p_4(w) \neq p_1 < 0$ be the limit point of the sequence $p_1(w_i)$, then $q(w, p_4) < r(w)$; on the other hand, the opposite bound $r(w) = q(w, p_1(w)) \leqslant q(w, p_3)$ follows from the bound $q(w_i, p_1(w)) \leqslant q(w_i, p_1(w_i))$.

All of the preceding arguments and bounds (4.11),(4.12) allow us to assert that no pair $u(w, v)$, $v_2(w)$ whatsoever can lead the position onto M earlier than at the instant $T(w)$. On the other hand, any pair $u_2(w, v)$, v leads the position onto M at precisely the instant $T(w)$. In spite of the fact that the study of the structure of the derivative of T for $w \in E \cap [r(w) = 0]$ can permit us to construct a control $u_3(w, v)$ which leads the position onto M earlier than at the instant $T(w)$ when $v \neq v_2(w)$, the answer to this question cannot improve the result when $v = v_2(w)$. In summary we can state a theorem.

Theorem 4.2. The pair of controls $u^\circ = u_2(w, v)$, $v^\circ = v_2(w)$ solves Problem 1 in the region $M^\circ[r(w) \geqslant 0;\ |x| > 0]$ with a time $T[u^\circ, v^\circ] = T(w)$.

Theorems 4.1 and 4.2 permit us to subdivide the whole space of positions into two sets

$$W^\circ(M) = M^\circ \cup M, \quad W_0(M) = M_0$$

in the first of which Problem 1 can be solved, while in the second, Problem 2.

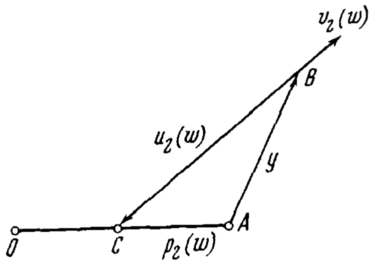

Fig. 1

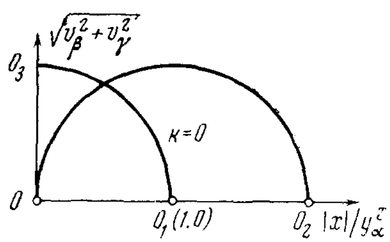

Fig. 2

5. Let us give a brief geometric interpretation of the results. Suppose that the plane of the diagram (Fig. 1) contains the vector $x \neq 0$ (OA) and the vector y (AB). The first player's optimal action is the realization of the impulse $u_2(w)$ (BC), so that the vector $y^{(1)}$ (AC) is antiparallel to vector x and equals the quantity $|p_2(w)|$ in absolute value, where $p_2(w)$ is the smallest root of the equation $q(w, p) = 0$. The second player's action is represented by the vector $v_2(w)$ antiparallel to vector $u_2(w)$

In what follows the inclusion $w^{(1)}(t) \in E \cap [r(w) = 0]$ is valid for $t > 0$ and the second player realizes the optimal control $v^\circ(w) = v_2(w)$; $w \in E \cap [r(w) = 0]$. The component of the vector, perpendicular to vector x, is arbitrary in direction in the plane normal to x, while its modulus tracks along the unit circle (O_1, O_3) when $k = 0$ and along the unit circle (O_2, O) when $k = 1$ (Fig. 2). The component v_α tracks along the segment (A_1O) when $k = 0$ (Fig. 3) and along the segment $(A_3O_1A_2)$ when $k = 1$.

Let us make a comparison with the one-dimensional analogs of the cases $k = 0$ and $k = 1$. When $k = 0$, $x > 0$ the obvious optimal actions of both players are the controls

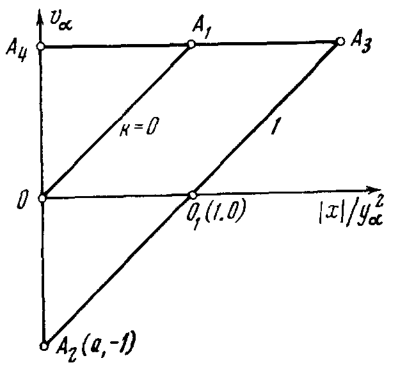

Fig. 3

$$u^\circ = -\mu\delta, \qquad v^\circ = +1 = v_\alpha$$

and contact takes place at the instant

$$T[u^\circ, v^\circ] = (1/2)\left[-(y_\alpha - \mu) - \sqrt{(y_\alpha - \mu)^2 - 4x}\right]$$

in the region $W^\circ\,[y_\alpha - \mu \leqslant 2\sqrt{x}]$. In case $k = 1$ [4] the optimal control is $v^\circ = v_\alpha = +1$ on the whole set W°. At positions of the one-dimensional analogs, which in the sense of the equalities

$$|x_{(1)}| = |x|, \quad y_\beta = 0, \quad y_\alpha = y_{(1)}, \quad \mu = \mu_{(1)}$$

duplicate the positions in the three-dimensional problem, the optimal time is strictly less than the optimal time in the three-dimensional problem. This fact is explained by the absence of a lateral maneuver in the one-dimensional problem.

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EXACT FORMULAS IN THE CONTROL PROBLEM OF CERTAIN SYSTEMS WITH AFTEREFFECT

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We examine the problem of the optimal control of linear systems with aftereffect and with quadratic performance criterion. We distinguish the class of such systems for which the corfficients of the optimal control and of the functional to be minimized are computed in explicit form.

1. Let there be given the controlled system

$$x^{\cdot}(t) = A(t)\,x(t) + B(t)\,x(t-h) + D(t)\,u(t), \qquad 0 \leqslant t \leqslant T \quad (1.1)$$

Here the vector $x(t)$ belongs to an Euclidean space R_n of dimension n, the control $u(t) \in R_m$, the constant $h \geqslant 0$, and A, B, D are given matrices with piecewise-